14.1 The Describing Function Method

The absolute value of $Y_f$ is the ratio of the amplitudes of the fundamental component after and before the nonlinearity. We can view $Y_f$ as an amplitude dependent gain. Writing $G$ in polar form

$$G(\omega) = |G(\omega)| e^{j\theta(\omega)}$$

we can restate the equations (14.2a) and (14.2b) in the form

$$Y_f(C)G(\omega) = -1$$

(14.7)

We see that the calculation of an oscillation consists of two steps.

1. Calculate $Y_f(C)$ from (14.6).
2. Solve (14.7).

Example 14.1: Ideal Relay

Consider the nonlinearity

$$f(e) = \begin{cases} 1 & \text{if } e > 0 \\ -1 & \text{if } e < 0 \end{cases}$$

(14.8)

i.e., an ideal relay. Using (14.5) gives

$$b(C) = \frac{1}{\pi} \int_0^\pi \sin \omega da + \frac{1}{\pi} \int_\pi^{2\pi} (-1) \sin \omega da = \frac{4}{\pi}$$

while (14.4) gives

$$a(C) = \frac{1}{\pi} \int_0^\pi \cos \omega da + \frac{1}{\pi} \int_\pi^{2\pi} (-1) \cos \omega da = 0$$

The describing function is thus real and given by

$$Y_f(C) = \frac{4}{\pi C}$$

(14.9)

The describing function calculated in this example is real. From (14.4) it follows that this is always the case for single-valued nonlinearities. Nonlinearities with complex valued describing functions are, e.g., backlash and hysteresis, see Examples 14.7 and 14.8 in Appendix 14A.

Software

In MATLAB the command nyquist, which draws a Nyquist diagram, is also useful in describing function calculations.

Appendix 14A: Some Describing Functions

Example 14.3: Cubic Gain

The cubic nonlinearity

$$f(u) = u^3$$

has the describing function

$$Y_f(C) = \frac{3C^2}{4}$$

Example 14.4: Relay with Dead Zone

A relay with dead zone is described by the relation

$$f(u) = \begin{cases} 1 & u > D \\ 0 & |u| \leq D \\ -1 & u < -D \end{cases}$$

and has the graph

$$Y_f(C) = \frac{4}{\pi C} \sqrt{1 - D^2/C^2}, \quad C > D$$
Appendix 14A: Some Describing Functions

Example 14.5: Saturation

A saturation is described by the relation

\[ f(u) = \begin{cases} 1 & u > 1 \\ u & [u] \leq 1 \\ -1 & u < -1 \end{cases} \]

and has the graph

\[ f(u) \]

\[ u \]

The describing function is

\[ Y_f(C) = \begin{cases} \frac{1}{2} \arcsin \frac{1}{C} + \frac{1}{2} \sqrt{1-C^{-2}} & C > 1 \\ 1 & C \leq 1 \end{cases} \]

Example 14.6: Dead Zone

A dead zone is described by the graph

\[ f(u) \]

\[ u \]

Its describing function is

\[ Y_f(C) = \begin{cases} \frac{H}{2} - \frac{H}{2B} \left( \arcsin(D/C) + \frac{B}{2} \sqrt{1 - \left(\frac{B}{C}\right)^2} \right) & C \geq D \\ 0 & C < D \end{cases} \]

Example 14.7: Relay with Hysteresis

A relay with hysteresis is described by the graph

\[ f(u) \]

\[ u \]

For \( u \)-values between \(-D\) and \( D \) the value of \( f(u) \) is not uniquely defined but is given by the following rule. If \( u \) has been greater than \( D \) then the value of \( f(u) \) remains at \( f(u) = H \) until \( u \) becomes less than \(-D\), when it is changed to \(-H\). If \( u \) has been less than \(-D\) the value \(-H\) remains until \( u \) becomes larger than \( D \). The describing function is well defined if \( C \geq D \), and is then given by

\[ \text{Re} Y_f(C) = \frac{4H}{\pi C} \sqrt{1 - D/C^2} \]

\[ \text{Im} Y_f(C) = -\frac{4D}{\pi C} \]

Example 14.8: Backlash

A backlash is described by the graph

\[ f(u) \]

\[ u \]

If \( u \) is increasing as a function of time, then \( f(u) \) is given by the sloping line on the right hand side, if \( u \) is decreasing by left hand one. If \( u \) changes from being increasing to being decreasing or vice versa, then \( f(u) \) is constant.
during the transition. This nonlinearity is an idealized description of what happens in, e.g., a pair of gears. If $C > D$ then the describing function is well defined and given by

\[
\text{Re}Y(f) = \frac{H}{\pi D} \left( \frac{\pi}{2} + \arcsin(1 - \frac{2D}{C}) + 2\left(1 - \frac{2D}{C}\right) \sqrt{\frac{D}{C}(1 - \frac{D}{C})} \right)
\]

\[
\text{Im}Y(f) = -\frac{4H}{\pi C} \left(1 - \frac{D}{C}\right)
\]