Review of stability analysis of LTI systems
Automatic control (10CFU) - AY 2015-2016

Simone Panza
Politecnico di Milano

31/03/2016
We have several tools for studying asymptotic stability of LTI systems:

- trace criterion
- characteristic polynomial coefficients
- characteristic polynomial roots
- Routh criterion
Let $tr(A)$ be the trace of a square matrix $A[n \times n]$

$$tr(A) = \sum_{i=1}^{n} a_{i,i}$$

If the system is asymptotically stable, then $tr(A) < 0$ (NC).
Characteristics polynomial coefficients

Let $p(s)$ be the characteristic polynomial

$$p(s) = \phi_0 s^n + \phi_1 s^{n-1} + \ldots + \phi_n \quad \phi_0 \neq 0$$

If the system is asymptotically stable, then $\phi_i$'s have all the same sign (NC).

Conversely, if $\phi_i$'s do not have the same sign, then the system is NOT asymptotically stable (but we cannot tell any further if it is unstable or simply stable).

If $n = 2$, then the condition becomes sufficient as well (NSC).

Example (only necessary, not sufficient condition)

$$p(s) = (s + a)(s^2 + b^2) = s^3 + as^2 + b^2s + ab^2,$$

$a, b > 0$ necessary condition is verified; still, the system is not asymptotically stable (2 poles on imaginary axis $s = \pm \mathbf{i}b$).
Let $p(s)$ be the characteristic polynomial

$$p(s) = \phi_0 s^n + \phi_1 s^{n-1} + \ldots + \phi_n \quad \phi_0 \neq 0$$

If the system is asymptotically stable, then $\phi_i, i = 0 \ldots n$ have all the same sign (NC).

Conversely, if $\phi_i$’s do not have the same sign, then the system is NOT asymptotically stable (but we cannot tell any further if it is unstable or simply stable).
Let $p(s)$ be the characteristic polynomial

$$p(s) = \phi_0 s^n + \phi_1 s^{n-1} + \ldots + \phi_n \quad \phi_0 \neq 0$$

If the system is asymptotically stable, then $\phi_i, i = 0 \ldots n$ have all the same sign (NC).
Conversely, if $\phi_i$'s do not have the same sign, then the system is NOT asymptotically stable (but we cannot tell any further if it is unstable or simply stable).
If $n = 2$, then the condition becomes sufficient as well (NSC).
Let $p(s)$ be the characteristic polynomial

$$p(s) = \phi_0 s^n + \phi_1 s^{n-1} + \ldots + \phi_n \quad \phi_0 \neq 0$$

If the system is asymptotically stable, then $\phi_i, i = 0 \ldots n$ have all the same sign (NC).

Conversely, if $\phi_i$'s do not have the same sign, then the system is NOT asymptotically stable (but we cannot tell any further if it is unstable or simply stable).

If $n = 2$, then the condition becomes sufficient as well (NSC).

**Example (only necessary, not sufficient condition)**

$$p(s) = (s + a)(s^2 + b^2) = s^3 + as^2 + b^2s + ab^2, \ a, b > 0$$

necessary condition is verified; still, the system is not asymptotically stable (2 poles on imaginary axis $s = \pm jb$)
Characteristic polynomial roots

Let \( s = s_i \) be the roots of the characteristic polynomial (i.e., the solutions of the characteristic equation \( p(s) = 0 \))

\[
p(s) = \phi_0 \prod_{i=1}^{n} (s - s_i)
\]

Stability can be studied based on the sign of \( \text{Re}(s_i) \)

- the system is asymptotically stable if and only if \( \text{Re}(s_i) < 0 \ \forall i = 1 \ldots n \)
- the system is unstable if \( \exists i | \text{Re}(s_i) > 0 \) or multiple eigenvalues exist on the imaginary axis (i.e., algebraic and geometric multiplicity is not the same)
- the system is (simply) stable if \( \text{Re}(s_i) \leq 0 \ \forall i \land \exists i | \text{Re}(s_i) = 0 \) and all the eigenvalue on the imaginary axis are simple (i.e., have same algebraic and geometric multiplicity).
Characteristic polynomial

\[ p(s) = \phi_0 s^n + \phi_1 s^{n-1} + \ldots + \phi_n \quad \phi_0 \neq 0 \]

Routh table

\[
\begin{array}{cccc}
\phi_0 & \phi_2 & \ldots & \phi_{n-1} \\
\phi_1 & \phi_3 & \ldots & \phi_n \\
\vdots \\
j_1 & j_2 & \ldots & j_i \\
k_1 & k_2 & \ldots & k_i \\
l_1 & l_2 & \ldots & l_i \\
\end{array}
\]

\[ l_i = -\frac{1}{k_1} \det \begin{pmatrix} j_1 & j_{i+1} \\ k_1 & k_{i+1} \end{pmatrix}, \quad k_1 \neq 0 \]

The system is asymptotically stable if and only if the coefficients of the first column have the same sign (NSC).
All of these tools can be used to study *asymptotic stability*:

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Necessary/sufficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trace</td>
<td>NC</td>
</tr>
<tr>
<td>Characteristic poly coefficients</td>
<td>NC *</td>
</tr>
<tr>
<td>Characteristic poly roots</td>
<td>NSC **</td>
</tr>
<tr>
<td>Routh</td>
<td>NSC</td>
</tr>
</tbody>
</table>

* becomes NSC if $n = 2$

** can also be used to assess simple stability or instability